Stability analysis of linear contraints nonholonomic systems based on conserved quantity

Xueting Liu [†] [‡] **Zhaolei Chen**[*†]

Abstract In this paper, we derive the augmented Birkhoff equation of linear contraints nonholonomic systems firstly. Base on a conserved quantity or a combination of some conserved quantities, we study the stability of linear contraints nonholonomic systems. Finally, a numerical example is provided to demonstrate the potential and effectiveness of the method.

Keywords conserved quantity; Birkhoff equation; linear contraints; stability.

1 Introduction

In this paper, we study the stability of linear contraints nonholonomic systems.

The stability of dynamical systems is one of the most basic issues in system theory. The theory of the stability of the nonholonomic control systems with linear contraints have attracted a lot of interest recently. The most complete contribution to the stability analysis of nonlinear systems was introduced by A. M. Lyapunov [1], the Lyapunov method is the most extensive analysis method currently. There are many stability results are obtained in the references, e.g., [2]. But it is very difficult to find a suitable Liapunov function.

Motivated by [3], in this paper, we derive the augmented Birkhoff equation of linear contraints nonholonomic systems firstly , construct a Lynpunov functional candidate by using a conserved quantity or a combination of some conserved quantities, and study the stability of linear contraints nonholonomic systems by the constructed Lynpunov functional candidate. Finally, a numerical example is provided to demonstrate the potential and effectiveness of the method.

2 Problem formulation and preliminaries

Lemma 1 [4] For the Birkhoff system

$$(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu})\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = 0(\mu,\nu = 1,2,\ldots,2n), \tag{1}$$

[*]Corresponding Author. E-mail address: liuxuet@l63.com.

[‡]School of Electrical and Electronic Engineering, Shandong University of Technology, Shandong, Zibo, 255049, People's Republic of China

if the infinitesimal transformations of group

$$t^* = t + \varepsilon_\alpha \xi_0^\alpha(t,a), \quad a^{\mu*}(t^*) = a^\mu(t) + \varepsilon_\alpha \xi_\mu^\alpha(t,a), \tag{2}$$

are the Noether quasi-symmetrical transformations, then the system possesses r linearly independent first integrals

$$I^\alpha = R_\mu \xi_\mu^\alpha - B\xi_0^\alpha + G^\alpha = C_\alpha \quad (\alpha = 1, 2, \dots, r), \tag{3}$$

where ε_α are infinitesimal parameters, ξ_0^α, ξ_μ^α, G^α are the generating functions and normalized function of the infinitesimal transformations respectively.

Lemma2 [4] If the infinitesimal transformations of group (5) satisfy the following r equetions

$$(\frac{\partial R_\mu}{\partial t}\dot{a}^\mu - \frac{B}{\partial t})\xi_0^\alpha + (\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{B}{\partial a^\mu})\xi_\mu^\alpha - B\dot{\xi}_0^\alpha + R_\mu \dot{\xi}_\mu^\alpha = -\dot{G}^\alpha \quad (\alpha = 1, 2, \dots, r), \tag{4}$$

then the transformations are quasi-symmetrical transformations of given system.

3 Main result

3.1 Birkhoff equation

Let $q = [q_1, q_2, \cdots, q_n]$ denote the generalized coordinate vector of nonholonomic system. The rheonomous affine kinematic model constraints are represented by analytical relations between the generalized coordinates and velocities $\dot{q} = [\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n]$, written as:

$$J(q)^T \dot{q} = 0, \tag{5}$$

where $J(q) \in R^{m\times n}$

Generally, the Lagrange function is:

$$L(q, \dot{q}, t) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(t, q), \tag{6}$$

where $M(q)$ is the $(n \times n)$ definite positive symmetric inertia matrix.

Using the Lagrange formalism, the dynamics of a mechanical system can be described by the following differential equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = J(q)\lambda \quad (s = 1, 2, \cdots, n), \tag{7}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T$ is the m-vector of Lagrange multipliers, is the control vector.

By [5], the multiplier λ_β can be solved as a function of t, $q, \dot{q}$ from the following equation

$$\sum_{\beta=1}^{m}\sum_{s=1}^{n}\sum_{l=1}^{n} M_{sl}^{-1} j_{l\gamma} j_{s\beta} \lambda_\beta$$

$$
\begin{aligned}
&= -\sum_{l=1}^{n} \frac{\partial a_\gamma}{\partial q_l}\dot{q}_l - \frac{\partial a_\gamma}{\partial t} + \sum_{l=1}^{n} j_{lr} \sum_{s=1}^{n} M_{sl}^{-1} \cdot \\
&\quad \left[\sum_{m=1}^{n}\sum_{k=1}^{n} [k,m,s]\dot{q}_k\dot{q}_m + \frac{\partial V}{\partial q_s} + \sum_{l=1}^{n} \frac{\partial M_{ks}}{\partial t}\dot{q}_k \right] \\
&= 0, \quad (\gamma = 1, 2, \cdots, m)
\end{aligned} \tag{8}
$$

where M_{sl} is an algebraic complement of M, and

$$
[k,m,s] = \frac{1}{2}\left(\frac{\partial M_{ks}}{\partial q_m} + \frac{\partial M_{ms}}{\partial q_k} - \frac{\partial M_{km}}{\partial q_s} \right). \tag{9}
$$

Take the generalized momentum

$$
p = \frac{\partial L}{\partial \dot{q}}, \tag{10}
$$

and $H = H(t, q, p)$ is the Hamiltonian

$$
H = p^T \dot{q} - L. \tag{11}
$$

From (8)-(11), we can express (7) as

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p}_s = -\frac{\partial H}{\partial q} + Q + u \quad (s = 1, 2, \cdots, n). \tag{12}
$$

where $Q = J(q)\lambda(q, p, t)$, $u = u'(q, p, t) = (u_1, u_2, \cdots, u_n)$

Let

$$
a^\mu = \begin{cases} q_\mu & (\mu = 1, 2, \ldots, n) \\ p_{\mu - n} & (\mu = n+1, n+2, \ldots, 2n) \end{cases} \tag{13}
$$

$$
R_\nu = \begin{cases} p_\nu & (\nu = 1, 2, \ldots, n) \\ 0 & (\nu = n+1, n+2, \ldots, 2n) \end{cases} \tag{14}
$$

$$
B(a) = H \tag{15}
$$

where $a = (a^1, a^2, \ldots, a^n)^T$. Then the Birkhoff equation is (13), (14), (15), where R_ν and B are Birkhoff' functions and Birkhofian respectively.

3.2 Stability

Let Lyapunov function $V(a, t)$ as a conserved quantity or a combination of some conserved quantities. Without loss of generality, let $V(a, t) = I^1$. We have the following conclusion:
Theorem 1. When $I^1 > 0$, $\dot{I}^1 \leq 0$, then the system is asymptotic stability.

4 An illustrative Example

Suppose rheonomous contraint is:

$$\dot{q}_1 + bt\dot{q}_2 - bq_2 = 0 \tag{16}$$

The kinetic energy and potential energy are

$$T = \frac{1}{2}(q_1^2 + q_2^2), \quad V = const \tag{17}$$

where b is constant.

The corresponding holonomic system of (16), (17) is

$$\begin{cases} \ddot{q}_1 = -\frac{1}{1+b^2t^2} \\ \ddot{q}_2 = -\frac{bt}{1+b^2t^2} \end{cases} \tag{18}$$

Then

$$\begin{cases} a^1 = q_1 \\ a^2 = q_2 \\ a^3 = \dot{q}_1 + \frac{1}{b}\arctan bt \\ a^4 = \dot{q}_2 + \frac{1}{2b}\ln(1+b^2t^2) \end{cases} \tag{19}$$

Equation (18) can be expressed as

$$\begin{cases} \dot{a}^1 = a^3 - \frac{1}{b}\arctan bt \\ \dot{a}^2 = a^4 - \frac{1}{2b}\ln(1+b^2t^2) \\ \dot{a}^3 = 0 \\ \dot{a}^4 = 0 \end{cases} \tag{20}$$

So we have

$$R_1 = a^3, \quad R_2 = a^4, \quad R_3 = 0, \quad R_4 = 0, \tag{21}$$

$$B = \frac{1}{2}[a^3 - \frac{1}{b}\arctan bt]^2 + \frac{1}{2}[a^4 - \frac{1}{2b}\ln(1+b^2t^2)]^2, \quad R_2 = a^4, \quad R_3 = 0, \quad R_4 = 0, \tag{22}$$

Let

$$\begin{cases} \xi_0^1 = 0 \\ \xi_1^1 = a^3, \xi_2^1 = 0, \xi_3^1 = 0, \xi_1^4 = 0, \\ G^1 = -\frac{1}{2}(a^3)^2 \end{cases} \tag{23}$$

So

$$\begin{cases} \Delta t = 0, \ (\Delta t)^* = 0 \\ \Delta a^1 = \varepsilon, \Delta a^2 = 0, \Delta a^3 = 0, \Delta a^4 = 0, \\ \Delta \dot{a}^\mu == 0 \end{cases} \tag{24}$$

thus (23) is Noether quasi-symmetrical transformation, and conserved quantity is

$$I^1 = \frac{1}{2}(a^3)^2, \tag{25}$$

Thus the system is asymptotic stability.

5 Conclusions

References

[1] A. M. Lyapunov.(1892) The General Problem of the Stability of Motion. Kharkov. Russia: Kharkov Mathematical Society.

[2] Zhengrong Xiang, Ronghao Wang, Qingwei Chen.(2011) Robust reliable control for uncertain switched nonlinear systems with time delay under asynchronous switching. Applied Mathematics and Computation, 217, pp.7725-7736.

[3] Mei Fengxiang, Wu Huibin.(2006) Zhang Yongfa. Stability with respect to partial variables for BirkhofF systems. Chinese Physics, 15(9), pp.1932-1934.

[4] Mei Fengxiang, Wu Huibin.(2009) Dynamics of Constrained Mechanical Systems. Beijing Institute of Technology Press, Beijing.

[5] Mei Fengxiang.(1983) The Foundations of mechanics of Nonholonomic System. Beijing Institute of Technology Press, Beijing.(In Chinese)